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Investigating Properties of the Information Relative Variance

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Introduction

Entropy is a central quantity in the field of thermodynamics that quantifies the amount of disorder present in the system. Nevertheless, it's limit lies in the fact that it has physical significance only for macroscopic system, when one has a large number of identical particles, so that fluctuations are negligible in the thermodynamics i.i.d limit. As we start to study thermodynamics of microscopic systems, one important quantity that comes into play is the variance of relative surprisal. This quantity allow us to capture the extent of thermal fluctuations in a system and is therefore important to addressing thermodynamics performance.

Relative Entropy & Relative Variance

Consider two quantum states ρ and σ with d-dimension, the entropy of ρ with respect to σ (also known as relative entropy) can be represented by:

$$S(\rho||\sigma) = -tr[\rho\,(\log(\rho) - \log(\sigma))]\ .$$

The relative variance between ρ and σ can be represented by:

$$V(\rho\|\sigma) = tr\left(\rho \log\left(\frac{\rho}{\sigma}\right)\right)^2 - S(\rho\|\sigma)^2$$

For classical case, the relative entropy and relative variance are defined via probability vectors **p** and **q**.

- Relative entropy $S(\mathbf{p}||\mathbf{q}) = \sum_i p_i \log\left(\frac{p_i}{q_i}\right)$
- Relative variance $V(\mathbf{p}||\mathbf{q}) = \sum_i p_i \left(\log\left(\frac{p_i}{q_i}\right)\right)^2 - S(\mathbf{p}||\mathbf{q})^2$

Maximum Relative Variance (Classical)

Consider 2 probability vectors **p** and **q**, and given that **q** = (q_1, q_2, …, q_d), w.l.o.g $q_1 \geq q_2 \geq \ldots \geq q_d$, the maximum relative variance V(**p**||**q**) is obtained for **p** = (r_2q_1, r_2q_2, …, r_2q_{d-1}, r_1q_d) where $\frac{p_d}{q_d} = r_1$, $\frac{p_i}{q_i} = r_2$ $\forall$ i = 1, 2, …, d-1.

The optimal form of relative variance in term of r_1 is:

$$V(\mathbf{p}||\mathbf{q}) = 4fr_1q_d\,(S(\mathbf{p}||\mathbf{q}) - 1) + (f+1)^2 - 2\,S(\mathbf{p}||\mathbf{q})(1+f)$$

Where $f = \frac{1}{2r_1q_d - 1}$

Maximum Relative Variance (Quantum)

Given ρ and σ as fixed qubit density matrices, with corresponding Bloch vector r and s respectively.

$$\rho = \frac{1}{2}(I + r_x\sigma_x + r_y\sigma_y + r_z\sigma_z)$$

$$\sigma = \frac{1}{2}(I + s_x\sigma_x + s_y\sigma_y + s_z\sigma_z)$$

Assume $s_x = s_y = 0$, and for a fixed value of r_z, the maximum $V(\rho||\sigma)$ over all possible density matrix ρ is obtained when $r_x = r_y = 0$. In other words, $V(\rho||\sigma)$ is maximized when ρ and σ commute.

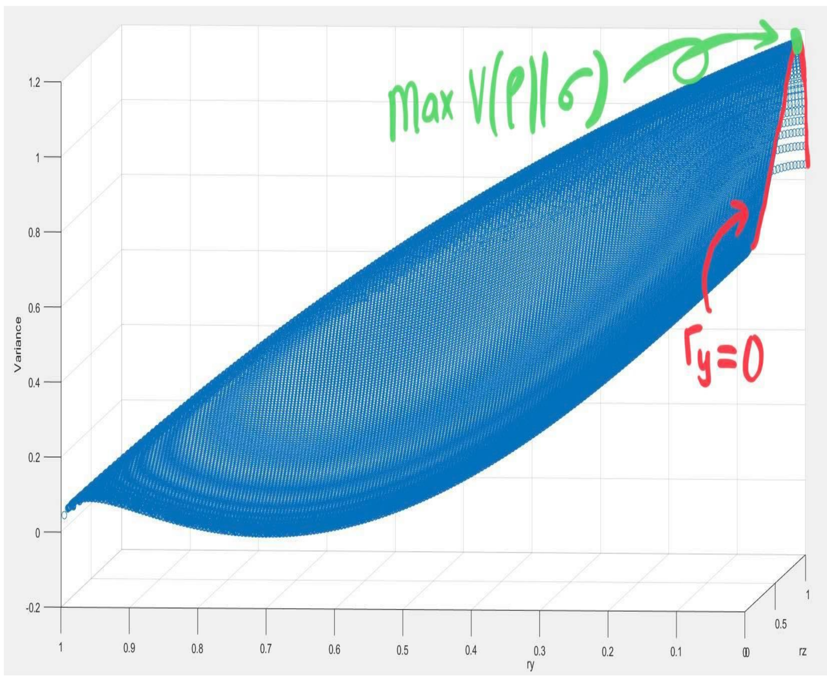

Figure1

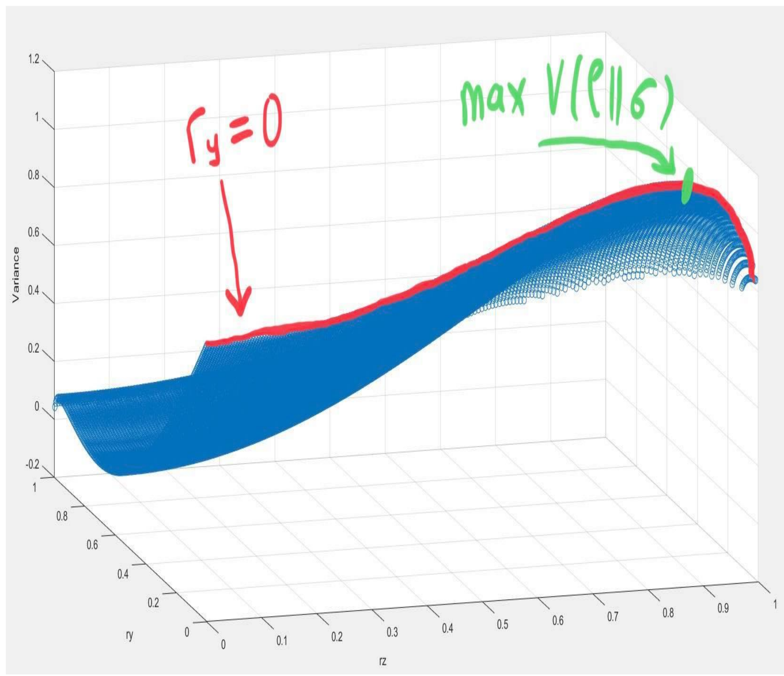

Figure 2

The finding above is based on the numeric observation from Figure 1 and Figure 2 where $r_z = 0.7$.

Conjecture

If we consider more generic case, in which d > 2 case, will the relative variance still maximize when two density matrices commute?

References

- P. Boes, N. H. Y. Ng, and H. Wilming, "The variance of relative surprisal as single-shot quantifier," arXiv:2009.08391.
- D. Reeb and M. M. Wolf, IEEE Trans. Inf. Theory 61, 1458 (2015).